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STABILITY OF FILTRATION OF A GAS–LIQUID MIXTURE

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The stability of steady regimes of filtration of a gas–liquid mixture at pressure lower than the saturation pressure is studied for the case of a nonmonotonic dependence of the relative phase permeability of the liquid on the gas saturation. It is shown that periodic self-oscillations can appear, and their evolution leads to deterministic chaos due to the appearance and destruction of quasiperiodic motions.

Usually, the isothermal filtration of a gas–liquid mixture at pressure lower than the saturation pressure is treated within the framework of the Musket–Meres classical model [1], and, in accordance with the experiments of Wyckoff and Botset [2], the relative phase permeabilities (RPP) for the liquid and gas are assumed to be monotonic functions of saturation. At the same time, Bolotov et al. [3] established that the flow rate of a gas–liquid mixture increases abnormally in the region of saturation pressure and decreases with further decrease in pressure, and this cannot be explained within the framework of the classical approach. Bolotov et al. [3] and Buevich [4] explain this effect by subcritical nucleation (accumulation of tiny gas bubbles in a porous medium and their subsequent removal), which leads to a reduction in the volume viscosity of the gas–liquid mixture. However, this reduction is only 10–15%, whereas the flow rate increases by a factor of 2.5–3 [3].

Shagapov [5] explains this effect by the "gas bearing" mechanism. However, he does not consider stability conditions for steady regimes and does not take into account that the slipping mechanism develops only in the saturation pressure region [3], where the porous medium is occupied primarily by a liquid with isolated gas bubbles (nucleation centers). Khasanov [6] studied the stability of motion of gas–liquid systems under subtransition conditions. He showed that for certain regimes of motion, periodic and stochastic self-oscillations can appear.

Churaev [7] showed experimentally that gas adsorption by microcapillaries weakens the adhesion of the liquid to the capillary surface and considerable slip of the liquid occurs even for an absorption layer 0.15 nm thick. As a result, the relative permeability of the liquid phase exhibits a nonmonotonic behavior. In this connection, it is of interest to study the effect of the nonmonotonic variation of the RPP of the liquid phase on the filtration characteristics of the gas–liquid mixture and, possibly, to obtain a quantitative estimate for this effect.

1. Equations of Unsteady Filtration. We consider the unsteady isothermal filtration of a gas–liquid mixture, employing the ideas of the theory of filtration of multicomponent systems [8]. For definiteness, we study filtration of oil with dissolved gas. Ignoring the solubility of the liquid in the gas, we write the continuity equations for the masses of the phases and components in the one-dimensional case:

$$\frac{\partial}{\partial t}(m\rho_1 s_1) + \frac{\partial}{\partial x}(m\rho_1 s_1 v_1) = -J_{12(2)}, \qquad \frac{\partial}{\partial t}(m\rho_2 s_2) + \frac{\partial}{\partial x}(m\rho_2 s_2 v_2) = J_{12(2)},$$
$$\frac{\partial}{\partial t}(m\rho_1 g s_1) + \frac{\partial}{\partial x}(m\rho_1 g s_1 v_1) = -J_{12(2)}, \qquad s_1 + s_2 = 1. \tag{1.1}$$

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Equations (1.1) are written for two phases (1 refers to the liquid phase and 2 refers to the gas phase) and for two component (1 corresponds to oil and 2 corresponds to the gas). Here s_i, v_i, and ρ_i are the saturation, the true flow velocity, and the true density of the ith phase, m is the porosity, g is the mass concentration the dissolved gas in the oil phase, $J_{ij(k)}$ is the rate of mass transfer per unit volume of the porous medium for degassing of the liquid, t is time, and x is the space coordinate. The velocity of the ith phase is defined by the Darcy law

$$ms_i v_i = -\frac{kf_i}{\mu_i}\frac{\partial p}{\partial x}, \tag{1.2}$$

where k is the absolute permeability of the medium, p is the pressure, and f_i and μ_i are the relative phase permeability and viscosity of the ith phase.

It is usual to employ an equilibrium model for filtration of a gas–liquid mixture, in which interphase exchange is described by the known isotherm of the solubility of the gas in the liquid. In particular, the phase concentration is described by the Henry solubility isotherm $g = g_e = \alpha p$ [$\alpha = \text{const}$ and $g_e(p)$ is the equilibrium phase concentration].

We assume that phase transitions are of a nonequilibrium nature. In this case, the mass concentration of the dissolved gas as a function of pressure is described by the following equation of the nonequilibrium model:

$$\frac{dg}{dt} = -\frac{g - g_e(p)}{\tau}, \tag{1.3}$$

where τ is the relaxation time, $dg/dt = \partial g/\partial t + v_1 \partial g/\partial x$ is the material derivative, and the velocity v_1 is determined from (1.2).

Equation (1.3) is in qualitative agreement with the well-known experimental data [9]. In the case of slow variation of the pressure in the system ($|dg/dt| \ll g/\tau$), where the process is close to an equilibrium process, we can set $dg/dt \approx 0$. Then, Eq. (1.3) reduces to the Henry law. This is also valid for $\tau = 0$, i.e., in the absence of a delay in the evolution or dissolution of the gas.

To simplify the analysis, we assume that the oil density and the viscosities of the phases are constant: ρ_1, μ_1, and $\mu_2 = \text{const}$. For the model problem, this choice is quite justified since the form of these functions does not affect the qualitative mechanisms of filtration of the gas–liquid mixture [8]. Assuming that the evolved gas is perfect, we set $\rho_2 = \rho_{2s} p/p_s$, where ρ_{2s} is the gas density at the cross section where gas evolution starts and p_s is the saturation pressure.

The determining relations for (1.1)–(1.3) depend on the phase state of the two-component medium. If $p > p_s$, the system is in the monophase state and $s_2 = 0$ and $J_{12(2)} = 0$. For $p \leqslant p_s$, there is a domain of local degassing, in which two-phase filtration takes place. The initial and boundary conditions are given by

$$\begin{gathered} t = 0, \quad x > 0, \quad p = p_0, \quad g = g_e^0, \\ t > 0, \quad x = 0, \quad p = p_0 > p_s, \quad s_2 = s = 0, \qquad t > 0, \quad x = L, \quad p = p_k < p_s, \end{gathered} \tag{1.4}$$

where s is the gas saturation.

System (1.1)–(1.4) describes the filtration of a gas–liquid mixture in the model of a porous material of length L and with pressure p_0 (higher than the saturation pressure) at the entrance and pressure p_k (lower than the saturation pressure) at the exit.

2. Relative Phase Permeability. For completion of system (1.1)–(1.4), it is necessary to specify the RPP coefficients f_i as functions of the gas saturation s. From an analysis of the experimental data of [3, 7, 10], it follows that the effect of slipping of the liquid is responsible for the nonmonotonic behavior of the relative permeability of the liquid phase.

Shagapov [5] obtained the relative liquid–phase permeability using the "gas bearing" model, according to which in the region where gas evolution begins, the gas phase is produced mainly in the layer adjacent to the pore walls in the porous medium. In the region of contact between the liquid and the solid surface of the pores there are most favorable conditions for evolution of the gas dissolved in the liquid (because of the

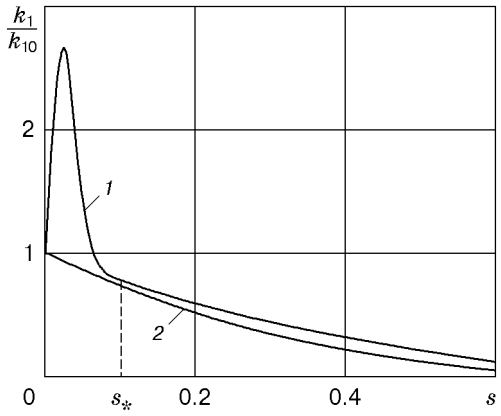

Fig. 1. Relative phase permeability versus gas saturation: curves 1 and 2 corresponds to calculations using the method proposed herein and the method of Wyckoff and Botset [2], respectively.

presence of nucleation centers). The wall layer of the liquid, which is saturated with gas nuclei, has lower viscosity than the central layer, and this results in slip flow.

The expression for the relative permeability of the liquid phase obtained in [5] includes viscosity. This implies the corresponding dependence of the liquid flow rate on the viscosity at the maximum point, which, however, does not agree with the experiment of [3]. For values $\mu_0 = \mu_2/\mu_1 < 0.1$, the maximum of the relative permeability of the oil phase can be attained at values of the gas saturation larger than 50%. The experiments and numerical calculations of [11] show that these values of the gas saturation correspond to a reduction in the pressure at the exit from the porous medium to $0.5p_s$ or lower. In this case, gas bubbles intensify degassing of the liquid in neighboring regions of the pores, and this facilitates the appearance of continuous gas-saturated pore channels [11]. Gas supply from the ambient liquid ensures the stability of the channels against capillary dispersion or, at least, fast recovery of their continuity. This leads to an increase in filtration resistance for the liquid phase and to a decrease in the liquid-flow rate.

Thus, the expression for the RPP function of the liquid given in [5] does not provide a fair fit to the experimental results of [3, 11], which imply that the maximum of this function must correspond to gas saturation values not exceeding the critical gas saturation s_*, which characterizes the saturation of the bound gas. Therefore, we assume that the phase permeability function for the liquid phase has the form shown in Fig. 1. The relative permeability is given by $f_i = k_i/k_{i0}$, where k_{i0} is the permeability measured for the phase i when it completely fills the pore volume and k_i is the phase permeability.

Experimental studies of the filtration of a gas–liquid mixture through a porous medium [6, 10] show that once the pressure difference becomes critical, steady filtration regimes lose stability and undamped time variations in the flow rate of the filtrated liquid are observed. Therefore, we study the stability of steady filtration regimes for the gas–liquid mixture taking into account the nonmonotonic behavior of the phase permeability for the liquid phase.

3. Linear Stability Analysis for Steady Regimes. We convert to dimensionless variables:

$$\bar p = \frac{p}{p_s}, \quad \bar x = \frac{x}{L}, \quad \bar t = \frac{t}{t_0}, \quad t_0 = \frac{\mu_1 L^2 m}{k p_s}, \quad \mu_0 = \frac{\mu_2}{\mu_1}, \quad \bar\rho_2 = \frac{\rho_2}{\rho_1} = \frac{\rho_{2s}}{\rho_1}\,\frac{p}{p_s} = \rho_0 \bar p,$$

$$\bar\tau = \frac{\tau}{t_0}, \quad g_e = \frac{g_s}{p_s}\,p = g_s \bar p, \quad g_s = \rho_0$$

(in what follows, the bar is omitted). The steady solutions $p^0(x)$, $s^0(x)$, and $g^0(x)$ are obtained by solving the system

$$\frac{\partial}{\partial x}\Big[(1-g)f_1(s)\,\frac{\partial p}{\partial x}\Big] = 0, \qquad \frac{\partial}{\partial x}\Big[\Big(g f_1 + f_2\,\frac{\rho_2}{\mu_0}\Big)\frac{\partial p}{\partial x}\Big] = 0,$$

$$\frac{f_1}{1-s}\,\frac{\partial p}{\partial x}\,\frac{\partial g}{\partial x} = -\frac{g(p)-g_s p}{\tau}$$

with appropriate boundary conditions. Steady solutions for the filtration of a gas–oil mixture were studied by Rosenberg et al. [8]. The functions $p^0(x)$, $s^0(x)$, and $g^0(x)$ are not identically constant. We linearize the initial equations (1.1) for the steady solution, setting $s \approx s^0 + s'$, $p \approx p^0 + p'$, and $g \approx g^0 + g'$ (s', p', and g' are small perturbations). For s', p', and g', we write the linearized equations

$$\partial_t \begin{pmatrix} s' \\ p' \\ g' \end{pmatrix} + A(x) \begin{pmatrix} s' \\ p' \\ g' \end{pmatrix} + B(x)\,\partial_x \begin{pmatrix} s' \\ p' \\ g' \end{pmatrix} + C(x)\,\partial_x^2 \begin{pmatrix} 0 \\ p' \\ 0 \end{pmatrix} = 0. \tag{3.1}$$

Here A, B, and C are matrices whose elements are evaluated for the steady solutions $s^0(x)$, $p^0(x)$, and $g^0(x)$.

Next, we study the long-wave solutions of system (3.1) using a simplified version of the Bubnov–Galerkin method [12]. We expand the solutions and the coefficients of system (3.1) into a Fourier series:

$$\begin{pmatrix} s' \\ p' \\ g' \end{pmatrix} = \begin{pmatrix} s_0(t) \\ p_0(t) \\ g_0(t) \end{pmatrix} + \sum_{k\neq 0} \exp(2\pi i k x) \begin{pmatrix} s_k(t) \\ p_k(t) \\ g_k(t) \end{pmatrix},$$

$$(A, B, C) = (A_0, B_0, C_0) + \sum_{k\neq 0} \exp(2\pi i k x)(A_k, B_k, C_k).$$

Substituting these expansions into the initial system (3.1), we obtain a system of ordinary differential equations for the coefficients of the Fourier series. Taking into account the convergence of the Fourier series (s_k, p_k, and $g_k \to 0$ as $|k| \to \infty$), we can consider a finite system of differential equations. This system is simplified by omitting all harmonics except for zeroth ones. This approach corresponds to solutions that vary slowly in x, i.e., long-wave perturbations. As a result, we obtain the system

$$d_t \begin{pmatrix} s_0 \\ p_0 \\ g_0 \end{pmatrix} = A \begin{pmatrix} s_0 \\ p_0 \\ g_0 \end{pmatrix}, \tag{3.2}$$

where the elements of the matrix A are calculated as the mean values in the steady state:

$$a_{11} = \int_0^1 \Big[\frac{g_s p^0 - g^0}{(1-g^0)\tau} - f_1''(s^0)s_x^0 p_x^0 - f_1'(s^0)p_{xx}^0\Big]dx, \quad a_{12} = -\int_0^1 \Big[\frac{1-s^0}{1-g^0}\,\frac{g_s}{\tau}\Big]dx,$$

$$a_{13} = \int_0^1 \Big[\frac{1-s^0}{(1-g^0)\tau} + \frac{1}{1-g^0}\,f_1(s^0)p_{xx}^0 + \frac{1}{1-g^0}\,f_1'(s_0)s_x^0 p_x^0\Big]dx,$$

$$a_{21} = \int_0^1 \Big[\frac{p^0}{s^0}\,f_1''(s^0)s_x^0 p_x^0 + \frac{p^0 - 1/\rho_0}{s^0(1-g^0)\tau}\,(g^0 - g_s p^0) + \frac{p^0}{\mu_0 s^0}\,f_2''(s^0)s_x^0 p_x^0$$

$$+ \frac{p^0}{s^0}\,f_1'(s^0)p_{xx}^0 + \frac{p^0}{\mu_0 s^0}\,f_2'(s^0)p_{xx}^0 + \frac{f_2'(s^0)}{\mu_0 s^0}\,(p_x^0)^2\Big]dx,$$

$$a_{22} = \int_0^1 \Big[\frac{g_s(p^0 - 1/\rho_0)(1-s^0) + (1-s^0)(g^0 - g_s p^0)}{s^0(1-g^0)\tau} + \frac{f_1'(s^0)}{s^0}\,s_x^0 p_x^0$$

$$+ \frac{f_1(s^0)}{s^0}\,p_{xx}^0 + \frac{f_2'(s^0)}{\mu_0 s^0}\,s_x^0 p_x^0 + \frac{f_2(s^0)}{\mu_0 s^0}\,p_{xx}^0\Big]dx,$$

$$a_{23} = -\int\limits_0^1 \Big[\frac{(p^0 - 1/\rho_0)(1-s^0)}{s^0(1-g^0)\tau} + \frac{p^0 f_1'(s^0)}{(1-g^0)s^0}\, s_x^0 p_x^0 + \frac{p^0 f_1(s^0)}{(1-g^0)s^0}\, p_{xx}^0$$

$$+ \frac{1}{\mu_0 s^0(1-g^0)}\,(p^0 f_2'(s^0) s_x^0 p_x^0 + f_2(s^0)(p_x^0)^2 + p^0 f_2(s^0) p_{xx}^0)\Big]dx,$$

$$a_{31} = \int\limits_0^1 \Big[\frac{f_1^0}{1-s^0}\, p_x^0 g_x^0 + \frac{g^0 - g_s p^0}{\tau}\Big]dx, \quad a_{32} = \frac{g_s}{\tau}, \quad a_{33} = -\frac{1}{\tau}.$$

We seek a solution of system (3.2) in the form

$$\begin{pmatrix} s_0 \\ p_0 \\ g_0 \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} \exp(\lambda t),$$

where the characteristic numbers λ are obtained from the equation

$$\begin{vmatrix} a_{11} - \lambda & a_{12} & a_{13} \\ a_{21} & a_{22} - \lambda & a_{23} \\ a_{31} & a_{32} & a_{33} - \lambda \end{vmatrix} = 0. \tag{3.3}$$

A numerical study of Eq. (3.3) was carried out for $p_0 = 1.25 p_s$, $p_s = 10$ MPa, and parameter values from the ranges $p_k = (0.5\text{–}0.9)p_s$, $\mu_0 = 0.001\text{–}0.010$ and $\tau = 0.001\text{–}1.0$. The analysis shows that the algebraic equation (3.3) always has three real roots. The number of zeros in the right half-plane is determined by the parameters $\Delta p = p_s - p_k$, μ_0, and τ. For $\Delta p \leqslant 0.5$ MPa and $\tau \leqslant 0.001$, the linear system is asymptotically stable for all t. However, in the range of values of the parameter Δp considered herein, the linear problem is always unstable since, under these conditions, Eq. (3.3) has at least one positive root. Moreover, if the values of the parameters Δp and τ increase, all roots increase and pass through zero, i.e., the instability increment increases. The character of the instability changes from the "saddle–node" type [one or two roots of (3.3) are negative] to the "unstable node" type (all three roots are positive). This conclusion is also valid for the complete linearized system (3.1), at least for solutions with sufficiently small Fourier amplitudes s_k, p_k, and g_k ($k \neq 0$). The instability in a linear approximation shows that the equilibrium given by $s^0(x)$, $p^0(x)$, and $g^0(x)$ is unstable. Self-oscillations can arise and develop when the parameter values change. In calculations with the RPP function for oil proposed in [5], all real roots of Eq. (3.3) are negative (stable node) and there are no oscillations in the linear system. This result is confirmed by the calculations presented in [5]. An analytical analysis of the complete equations (1.1) is difficult because of their complexity. Therefore, we restrict ourselves to numerical calculations.

4. Calculation Results. To analyze the filtration process in the instability region, we solve system (1.1)–(1.3) numerically using a conservative finite-difference scheme. The pressure distribution in the bed is calculated from the condition of conservation of mass for the flow using the sweep method for an implicit four-point scheme. The values obtained are used to calculate the saturation and phase concentration. The coefficients for the lower layer are used to exclude nonlinearity. The relative permeability of the liquid phase is chosen in such a way that the maximum of the function corresponds to the gas saturation value s for which the evolved gas is entirely adsorbed by the pore walls. The gas evolved acquires mobility, forming a bound phase, at mean gas saturation $s = s_* \approx 0.1$. Therefore, the RPP of the liquid phase decreases to values smaller than k_{10}.

Steps in time Δt and coordinate h were chosen from the stability condition for the finite-difference scheme, so that their further reduction by a factor of two or more did not change the qualitative and quantitative picture of the calculations. An analysis of the solution on various spatial grids showed that the stability and sufficient accuracy of the calculations are ensured for $h \leqslant 0.01$ and $\Delta t \leqslant 0.0000025$. Multivariant

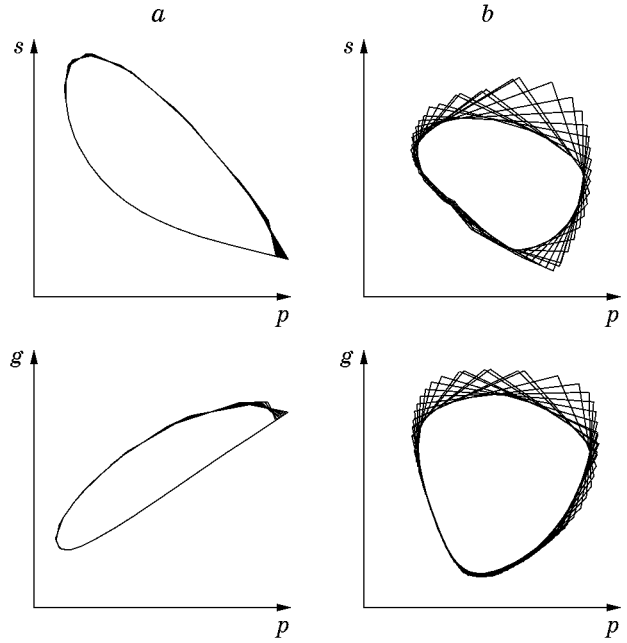

Fig. 2. Evolution of the attractor of oscillatory motion for $\Delta p = 3$ MPa, $\tau = 0.01$, and $\mu_0 = 0.01$ (a) and $\Delta p = 5$ MPa, $\tau = 0.1$, and $\mu_0 = 0.001$ (b).

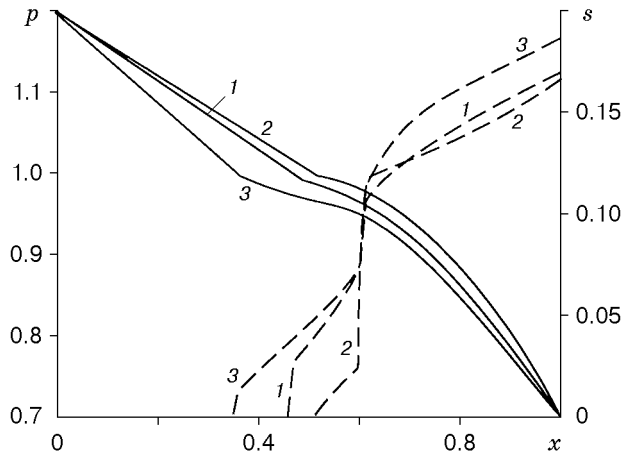

Fig. 3. Distributions of the pressure p (solid curves) and gas saturation s (dashed curve) over the length of the model for $t = 1.2$ (1) 2.4 (2), and 3.6 (3).

calculations of the pressure and saturation fields were performed for the following hydrodynamic parameters of the phases: $\mu_0 = 0.001$–0.010, $\tau = 0.01$–1, $p_k = (0.5$–$0.9)p_s$, $p_0 = (1.1$–$1.3)p_s$, $\mu_2 = 0.01$ mPa $\cdot$ sec, $g_s = 0.11$, $\rho_1 = 700$ kg/m^3, $\rho_2 = 68$ kg/m^3, $L = 1$ m, $k = 0.00095$ μm^2, and $m = 0.2$.

The calculations show that in the domain of filtration of the gas–oil mixture, time-periodic changes of the pressure and saturation occur. The stability region is determined by the parameter μ_0 and the pressure difference $\Delta p = p_s - p_k$. With $\mu_0 \geqslant 0.01$, the motion is stable for $\Delta p_* \leqslant 0.5$ MPa. As μ_0 decreases to 0.001, the critical value Δp_* increases to 1 MPa. If $0.001 < \mu_0 < 0.01$ and $\Delta p > p_*$, a periodic self-oscillation regime arises, which is stable within the above-mentioned range of the parameter μ_0. Growth in τ with fixed pressure difference decreases the mean gas saturation. With increase in the pressure difference at fixed τ, the oscillation frequency and amplitude and the mean gas saturation increase. For $\mu_0 \leqslant 0.001$, an increase in the pressure difference Δp to 3 MPa leads to loss of stability of the limiting cycle. As Δp and relaxation time τ increase, new types of perturbations appear and quasiperiodic motion arises. The evolution of the attractor of the oscillatory motion (projections onto the phase planes $p \sim s$ and $p \sim g$) is shown in Fig. 2.

With further increase in τ, the quasiperiodic oscillations fail and become chaotic ($\tau_{\text{k.p.}} = 1$). It follows from the numerical results that the transition to chaos in the system considered occurs via failure of quasiperiodic motion.

Figure 3 shows distributions of the gas saturation s and pressure p over the length of the model of a porous medium for various times for $p_k = 7$ MPa, $\Delta p = 3$ MPa, and $\tau = 0.01$. One can see two "jumps" in the gas saturation distribution. For the first of them, the gas saturation is lower than the equilibrium value and the evolved gas is motionless, and the second corresponds to the value $s \approx s_*$, for which the gas acquires mobility. Periodic oscillations of the pressure and saturation produce corresponding variations in the dimensions of the degassing region. Therefore, near the phase-transition front (at $p = p_s$) there is a motionless gas-phase region. Precisely this region is responsible for nonmonotonic changes in the phase permeability of the liquid phase, which, ultimately, lead to unstable filtration regimes.

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